Repetitions in infinite rich words

A computational approach

Aseem Baranwal, Jeffrey Shallit

School of Computer Science
University of Waterloo
Canada

Email : aseemrb@gmail.com
Web : https://aseemrb.me

Overview

1. Introduction
2. Results over binary alphabet
3. Repetition threshold
4. Future work

Rich words

- A finite word w is palindrome-rich, or simply *rich* if it contains $|w|$ nonempty distinct palindromic factors.
- An infinite word is *rich* if all of its factors are rich.

Examples

- The word 00010110 is rich – contains 8 distinct nonempty palindromes: 0, 00, 000, 1, 010, 101, 11, 0110.
- The word 00101100 is not rich – only 7 distinct palindromes.

Repetition Threshold

- Repetition threshold can be studied for a limited class of infinite words – e.g., Rampersad et al. considered the class of balanced words.
- We study the class of infinite palindrome-rich words.

- Repetition threshold can be studied for a limited class of infinite words – e.g., Rampersad et al. considered the class of balanced words.
- We study the class of infinite palindrome-rich words.

Problem

What is the repetition threshold for infinite rich words over a k-letter alphabet?

Known results

Known results

- Pelantová and Starosta (2013) showed that every infinite rich word contains a square, implying that the repetition threshold is greater than 2.

Known results

- Pelantová and Starosta (2013) showed that every infinite rich word contains a square, implying that the repetition threshold is greater than 2.
- Vesti gave upper and lower bounds on the length of the longest square-free rich words (2017). He also proposed the open problem of determining the repetition threshold.

Known results

- Pelantová and Starosta (2013) showed that every infinite rich word contains a square, implying that the repetition threshold is greater than 2.
- Vesti gave upper and lower bounds on the length of the longest square-free rich words (2017). He also proposed the open problem of determining the repetition threshold.

Our contribution

Known results

- Pelantová and Starosta (2013) showed that every infinite rich word contains a square, implying that the repetition threshold is greater than 2.
- Vesti gave upper and lower bounds on the length of the longest square-free rich words (2017). He also proposed the open problem of determining the repetition threshold.

Our contribution

- We make the first progress on Vesti's problem by constructing a binary word that achieves the repetition threshold.

Known results

- Pelantová and Starosta (2013) showed that every infinite rich word contains a square, implying that the repetition threshold is greater than 2.
- Vesti gave upper and lower bounds on the length of the longest square-free rich words (2017). He also proposed the open problem of determining the repetition threshold.

Our contribution

- We make the first progress on Vesti's problem by constructing a binary word that achieves the repetition threshold.
- Our approach is automated and computation-based (`Walnut`).

Morphisms

$$\phi: \quad \begin{aligned} 0 &\mapsto 01 \\ 1 &\mapsto 02 \\ 2 &\mapsto 022 \end{aligned} \qquad\qquad \tau: \quad \begin{aligned} 0 &\mapsto 0 \\ 1 &\mapsto 01 \\ 2 &\mapsto 011 \end{aligned}$$

Theorem

The infinite word $\mathbf{r} = \tau(\phi^\omega(0)) = 00100110010\ldots$ is rich, and has the critical exponent $2 + \frac{\sqrt{2}}{2}$.

Morphisms

$$\phi:\quad \begin{aligned} 0 &\mapsto 01 \\ 1 &\mapsto 02 \\ 2 &\mapsto 022 \end{aligned} \qquad\qquad \tau:\quad \begin{aligned} 0 &\mapsto 0 \\ 1 &\mapsto 01 \\ 2 &\mapsto 011 \end{aligned}$$

Theorem

The infinite word $\mathbf{r} = \tau(\phi^\omega(0)) = 00100110010\ldots$ is rich, and has the critical exponent $2 + \frac{\sqrt{2}}{2}$.

- We conjectured this exponent to be the repetition threshold among the class of infinite rich words over $\Sigma_2 = \{0, 1\}$.
- Currie et al. have resolved our conjecture.

Observation

The lengths $L_i = |\tau(\phi^i(0))|$ follow the recurrence relation:
$L_0 = 1$, $L_1 = 3$, and $L_i = 2L_{i-1} + L_{i-2}$ for all $i \geq 2$.

This suggests that the word $\mathbf{r}$ might be Pell-automatic.

Constructing The Automaton

Observation

The lengths $L_i = |\tau(\phi^i(0))|$ follow the recurrence relation: $L_0 = 1$, $L_1 = 3$, and $L_i = 2L_{i-1} + L_{i-2}$ for all $i \geq 2$.

This suggests that the word $\mathbf{r}$ might be Pell-automatic.

- We require an automaton producing the word $\mathbf{r}$ for `Walnut` to understand predicates involving $\mathbf{r}$.
- We use the adder automaton given by Baranwal and Shallit to work in Walnut with predicates involving the word.

Constructing the automaton

- We construct the automaton producing the word **r** using the L* algorithm by Angluin [1] to learn regular sets with queries.
- An induction based proof verifies that this automaton produces the same word as $\tau(\phi^\omega(0))$.

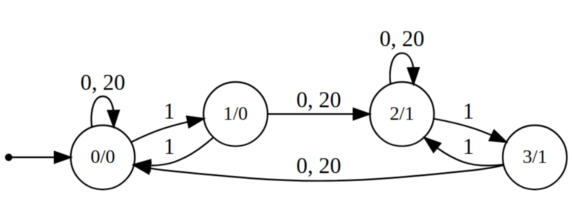


Figure 1: Automaton for the infinite word **r**.

Notation

R denotes the automaton for the word r indexed from 0.

Notation

R denotes the automaton for the word r indexed from 0.

Procedure overview

1. Construct automata for a set of fundamental predicates.

Notation

R denotes the automaton for the word r indexed from 0.

Procedure overview

1. Construct automata for a set of fundamental predicates.
2. Use them to construct a predicate for palindromic richness.

Notation

R denotes the automaton for the word r indexed from 0.

Procedure overview

1. Construct automata for a set of fundamental predicates.
2. Use them to construct a predicate for palindromic richness.
3. Show that the predicate is true for all inputs.

Length-n factors of R starting at indices i and j are equal.

FactorEq(i, j, n)

$\forall k\ (k < n) \implies \mathsf{R}[i + k] = \mathsf{R}[j + k]$

Length-n factors of R starting at indices i and j are equal.

FactorEq(i, j, n)
$\forall k\ (k < n) \implies \mathrm{R}[i + k] = \mathrm{R}[j + k]$

Length-n factor of R starting at index i is a palindrome.

Palindrome(i, n)
$\forall j\ \forall k\ (k < n) \implies \mathrm{R}[i + k] = \mathrm{R}[n - 1 - k]$

Length-n factors of R starting at indices i and j are equal.

FactorEq(i, j, n)
$\forall k\ (k < n) \implies \mathrm{R}[i + k] = \mathrm{R}[j + k]$

Length-n factor of R starting at index i is a palindrome.

Palindrome(i, n)
$\forall j\ \forall k\ (k < n) \implies \mathrm{R}[i + k] = \mathrm{R}[n - 1 - k]$

The word $\mathrm{R}[i..i + m - 1]$ occurs in the word $\mathrm{R}[j..j + n - 1]$.

Occurs(i, j, m, n)
$(m \leq n) \land (\exists k\ (k + m \leq n) \land \texttt{FactorEq}(i, j + k, m))$

Fundamental predicates

Length-n factors of R starting at indices i and j are equal.

$\texttt{FactorEq}(i, j, n)$	67 states
$\forall k\ (k < n) \implies \mathrm{R}[i+k] = \mathrm{R}[j+k]$	

Length-n factor of R starting at index i is a palindrome.

$\texttt{Palindrome}(i, n)$	27 states
$\forall j\ \forall k\ (k < n) \implies \mathrm{R}[i+k] = \mathrm{R}[n-1-k]$	

The word $\mathrm{R}[i..i+m-1]$ occurs in the word $\mathrm{R}[j..j+n-1]$.

$\texttt{Occurs}(i, j, m, n)$	1715 states
$(m \leq n) \wedge (\exists k\ (k+m \leq n) \wedge \texttt{FactorEq}(i, j+k, m))$	

Fact

An infinite word is rich if and only if all its factors are rich.
We could also look at only the prefixes instead of all factors.

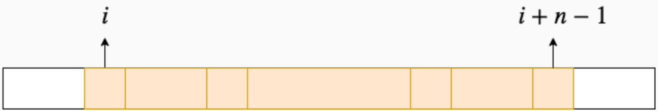
Fact

An infinite word is rich if and only if all its factors are rich.
We could also look at only the prefixes instead of all factors.

Theorem

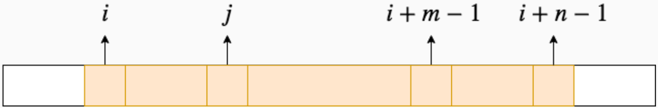
(Glen et al.) A finite word w is rich if and only if every prefix of w has a unioccurrent palindromic suffix.

Figure 2: Constructing the predicate `RichFactor`(i, n).



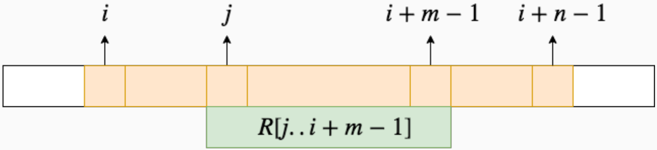
`RichFactor`(i, n):

Figure 2: Constructing the predicate $\texttt{RichFactor}(i, n)$.



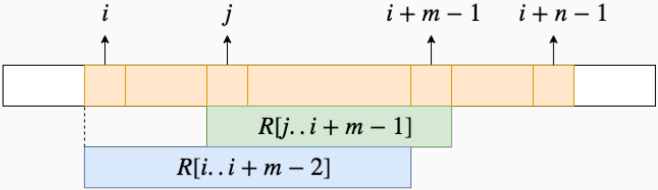
$\texttt{RichFactor}(i, n)$: $\forall m, (1 \leq m < n) \implies (\exists j, (i \leq j < i+m) \wedge$

Figure 2: Constructing the predicate $\texttt{RichFactor}(i, n)$.



$\texttt{RichFactor}(i, n)$: $\forall m, (1 \leq m < n) \implies (\exists j, (i \leq j < i + m) \wedge$
$\texttt{Palindrome}(j, i + m - j))$

Figure 2: Constructing the predicate `RichFactor`(i, n).



$$\texttt{RichFactor}(i,n)\colon \forall m, (1 \le m < n) \implies (\exists j, (i \le j < i+m) \land \texttt{Palindrome}(j, i+m-j) \land \neg\texttt{Occurs}(j, i, i+m-j, m-1))$$

To determine if $\mathbf{r}$ is rich, we check if all its prefixes are rich.

R_is_Rich

$\forall n\ \texttt{RichFactor}(0, n)$.

- In `Walnut`, this predicate evaluates to true.
- Conclusion – The infinite word $\mathbf{r}$ is rich.

Problem

It is difficult to write a first-order predicate to determine the critical exponent because it is an irrational number.

Problem

It is difficult to write a first-order predicate to determine the critical exponent because it is an irrational number.

Solution

Compute the critical exponent as the limit of a monotonic expression for exponents.

Problem

It is difficult to write a first-order predicate to determine the critical exponent because it is an irrational number.

Solution

Compute the critical exponent as the limit of a monotonic expression for exponents.

Overview

- Compute periods of high powers ($\geq 5/2$).
- Compute the maximal lengths associated with the high-power periods above.

Computing the critical exponent

First, compute periods p such that the word r has factors with period p and exponent $> 5/2$.

HighPowerPeriods(p)
$(p \geq 1) \wedge (\exists i \forall j\ (2j \leq 3p) \implies \mathrm{R}[i+j] = \mathrm{R}[i+j+p])$.

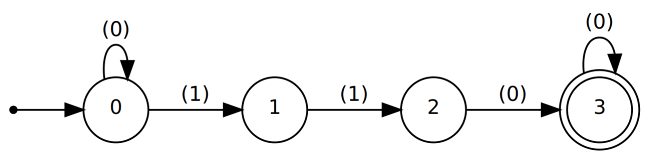


Figure 3: Automaton for the predicate HighPowerPeriods.

Compute pairs (n, p) such that $\mathbf{r}$ has a factor of length $n + p$ with period p, which cannot be extended with the same period.

MaximalReps(n, p) — 17 states

$\exists i (\forall j \ (j < n) \implies \mathrm{R}[i + j] = \mathrm{R}[i + j + p]) \wedge (\mathrm{R}[i + n] \neq \mathrm{R}[i + n + p]).$

Compute pairs (n, p) such that $\mathbf{r}$ has a factor of length $n + p$ with period p, which cannot be extended with the same period.

`MaximalReps`(n, p) — 17 states

$\exists i (\forall j\ (j < n) \implies \mathrm{R}[i + j] = \mathrm{R}[i + j + p]) \wedge (\mathrm{R}[i + n] \neq \mathrm{R}[i + n + p])$.

Compute pairs (n, p) where p satisfies `HighPowerPeriods` and $n + p$ is the longest length of any factor with period p.

`HighestPowers`(n, p)

`HighPowerPeriods`$(p) \wedge$
`MaximalReps`$(n, p) \wedge$
$(\forall m$ `MaximalReps`$(m, p) \implies m \leq n)$.

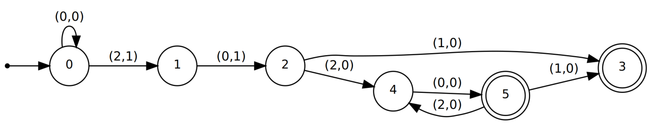


Figure 4: Automaton for the predicate HighestPowers.

Critical exponent

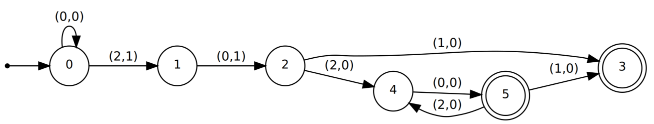


Figure 4: Automaton for the predicate `HighestPowers`.

This automaton accepts pairs (n, p) of the following formats:

1. $\binom{0}{0}^* \binom{2}{1} \binom{0}{1} \binom{1}{0}$,
2. $\binom{0}{0}^* \binom{2}{1} \binom{0}{1} \binom{2}{0} \binom{0}{0} \left\{ \binom{2}{0} \binom{0}{0} \right\}^*$,
3. $\binom{0}{0}^* \binom{2}{1} \binom{0}{1} \binom{2}{0} \binom{0}{0} \left\{ \binom{2}{0} \binom{0}{0} \right\}^* \binom{1}{0}$.

- **Case 1** corresponds to: $n = (201)_P = 11$ and $p = (110)_P = 7$.

$$e = \frac{n+p}{p} = \frac{18}{7} \approx 2.57.$$

- **Case 2** corresponds to:

$$n = 2 \sum_{1 \le i \le k} P_{2k} = P_{2k+1} - 1, \quad p = P_{2k} + P_{2k-1}$$

- **Case 3** corresponds to:

$$n = 1 + 2 \sum_{1 \le i \le k} P_{2k+1} = P_{2k+2} - 1, \quad p = P_{2k+1} + P_{2k}$$

Substituting $m = 2k - 1$ in case 2, and $m = 2k$ in case 3, we notice that the expressions for the exponent coincide:

$$\begin{aligned} e &= \frac{P_{m+2} + P_{m+1} + P_m - 1}{P_{m+1} + P_m} \\ &= 2 + \frac{P_{m+1} - 1}{P_{m+1} + P_m}. \end{aligned}$$

This expression is increasing with m, and tends to $2 + \sqrt{2}/2$ as $m \to \infty$. Thus, the critical exponent,

$$E(\mathrm{r}) = 2 + \frac{\sqrt{2}}{2}.$$

- With backtracking search, we had found that

$$2.700 \leq RRT(2) \leq 2 + \frac{\sqrt{2}}{2} \doteq 2.707,$$

and conjectured that the upper bound is exact.

- Used the data structure EERTREE given by Rubinchik and Shur [5] to efficiently verify richness.

- With backtracking search, we had found that

$$2.700 \leq RRT(2) \leq 2 + \frac{\sqrt{2}}{2} \doteq 2.707,$$

  and conjectured that the upper bound is exact.
- Used the data structure EERTREE given by Rubinchik and Shur [5] to efficiently verify richness.
- Recently, Currie, Mol, and Rampersad have resolved our conjecture.

Morphisms f, g and h

$$f\colon\ \begin{aligned}0&\mapsto 0\\1&\mapsto 01\\2&\mapsto 011\end{aligned}\qquad g\colon\ \begin{aligned}0&\mapsto 011\\1&\mapsto 0121\\2&\mapsto 012121\end{aligned}\qquad h\colon\ \begin{aligned}0&\mapsto 01\\1&\mapsto 02\\2&\mapsto 022\end{aligned}$$

Lemma (Currie et al.)

The critical exponent of $f(g(h^\omega(0)))$ is at least $2+\sqrt{2}/2$.

Proof by Currie et al.

Theorem (Currie, Mol, Rampersad)

The repetition threshold for infinite rich words over the alphabet $\Sigma_2 = \{0, 1\}$ is $2 + \sqrt{2}/2$.

Proof sketch

Theorem (Currie, Mol, Rampersad)

The repetition threshold for infinite rich words over the alphabet $\Sigma_2 = \{0, 1\}$ is $2 + \sqrt{2}/2$.

Proof sketch

- If $w \in \Sigma_2^\omega$ is an infinite rich word avoiding powers $\geq 14/5$ then

Theorem (Currie, Mol, Rampersad)

The repetition threshold for infinite rich words over the alphabet $\Sigma_2 = \{0, 1\}$ is $2 + \sqrt{2}/2$.

Proof sketch

- If $w \in \Sigma_2^\omega$ is an infinite rich word avoiding powers $\geq 14/5$ then
  - w contains all factors of $f(h^\omega(0))$, or

Theorem (Currie, Mol, Rampersad)

The repetition threshold for infinite rich words over the alphabet $\Sigma_2 = \{0, 1\}$ is $2 + \sqrt{2}/2$.

Proof sketch

- If $w \in \Sigma_2^\omega$ is an infinite rich word avoiding powers $\geq 14/5$ then
  - w contains all factors of $f(h^\omega(0))$, or
  - w contains all factors of $f(g(h^\omega(0)))$.

Theorem (Currie, Mol, Rampersad)

The repetition threshold for infinite rich words over the alphabet $\Sigma_2 = \{0, 1\}$ is $2 + \sqrt{2}/2$.

Proof sketch

- If $w \in \Sigma_2^\omega$ is an infinite rich word avoiding powers $\geq 14/5$ then
  - w contains all factors of $f(h^\omega(0))$, or
  - w contains all factors of $f(g(h^\omega(0)))$.
- Our word $\mathbf{r} = f(h^\omega(0))$ has critical exponent $2 + \sqrt{2}/2$.

Theorem (Currie, Mol, Rampersad)

The repetition threshold for infinite rich words over the alphabet $\Sigma_2 = \{0, 1\}$ is $2 + \sqrt{2}/2$.

Proof sketch

- If $w \in \Sigma_2^\omega$ is an infinite rich word avoiding powers $\geq 14/5$ then
  - w contains all factors of $f(h^\omega(0))$, or
  - w contains all factors of $f(g(h^\omega(0)))$.
- Our word $\mathbf{r} = f(h^\omega(0))$ has critical exponent $2 + \sqrt{2}/2$.
- By Lemma, the critical exponent of $f(g(h^\omega(0)))$ is $\geq 2 + \sqrt{2}/2$.

Proof by Currie et al.

Theorem (Currie, Mol, Rampersad)

The repetition threshold for infinite rich words over the alphabet $\Sigma_2 = \{0, 1\}$ is $2 + \sqrt{2}/2$.

Proof sketch

- If $w \in \Sigma_2^\omega$ is an infinite rich word avoiding powers $\geq 14/5$ then
  - w contains all factors of $f(h^\omega(0))$, or
  - w contains all factors of $f(g(h^\omega(0)))$.
- Our word $\mathbf{r} = f(h^\omega(0))$ has critical exponent $2 + \sqrt{2}/2$.
- By Lemma, the critical exponent of $f(g(h^\omega(0)))$ is $\geq 2 + \sqrt{2}/2$.

The repetition threshold for binary rich words, James D. Currie, Lucas Mol, Narad Rampersad, Arxiv preprint: https://arxiv.org/abs/1908.03169.

Fact (Edita Pelantová)

Our word $\mathbf{r}$ is a complementary symmetric Rote word [4], and hence by the works of Massé, Pelantová and others [2, 3], it follows that $\mathbf{r}$ is rich.

- Repetition threshold for larger alphabets. Our backtracking search shows that $RRT(3) \geq 9/4$.
- Construct Rote words associated with Sturmian substitutions over larger alphabets (k) – check if they achieve $RRT(k)$.

D. Angluin.
Learning regular sets from queries and counterexamples.
Information and Computation, 75(2):87–106, 1987.

A. Blondin Massé, S. Brlek, S. Labbé, and L. Vuillon.
Palindromic complexity of codings of rotations.
Theoret. Comput. Sci., 412:6455–6463, 2011.

E. Pelantová and Š. Starosta.
Constructions of words rich in palindromes and pseudopalindromes.
Discrete Math. & Theoret. Comput. Sci., 18:Paper #16, 2016.
Available at `https://dmtcs.episciences.org/2202`.

G. Rote.
Sequences with subword complexity $2n$.
J. Number Theory, 46:196–213, 1994.

M. Rubinchik and A. M. Shur.
EERTREE: An efficient data structure for processing palindromes in strings.
In Z. Lipták and W. F. Smyth, editors, *Combinatorial Algorithms*, pages 321–333, Cham, 2016. Springer International Publishing.

Thank you.